

\mathcal{C} , \mathcal{P} and \mathcal{T} operations and classical point charged particle dynamics

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Abstract

The action of parity inversion, time inversion and charge conjugation transformations on several differential equations for a classical point charged particle are described. We consider an observable quantity Δ_q which is sensitive to deviations from the standard Lorentz force equation. We argue that Δ_q could be observable with current or near future technology.

Introduction

It is of high relevance for the developments of electrodynamics to have a consistent theory of radiation reaction of classical particles. Not only this is important from a theoretical perspective, but also for applications in accelerator science and laser-plasma interactions. Indeed, modern applications on those fields are reaching dynamical regimens where radiation reaction effects is of relevance [2, 16].

However, the theory of radiation reaction of charged particles is severely challenged by the fact that are very few physical systems able to probe an experimental signal from individual radiation reaction effects. Among them are the *Penning trap* [14, 15] and high intensity laser-plasma systems [16]. Therefore, any other possible observable effect for the radiation reaction will be of relevance in the discussion of the correct model of radiation reaction.

In this letter we show how the parity inversion \mathcal{P} , the time inversion \mathcal{T} and the charge conjugation \mathcal{C} operations act on several equations of motion of point charged particles with radiation reaction. We show how the study of these operations provides a strong insight in a new quantity which is sensitive to the radiation reaction and that it is observable with current technology.

We first define the (classical) operations \mathcal{P} , \mathcal{T} and \mathcal{C} acting on the electromagnetic fields, that will be solutions of Maxwell's theory. Then we show how the operations \mathcal{P} , \mathcal{T} and \mathcal{C} act on several equations of motion. After this, we consider the *symmetrized four-acceleration* $\Delta_q^\mu = \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu$ for each of the models. Δ_q^μ measures how different is the dynamics of a particle and anti-particle under the influence of an external electromagnetic field. It is highly sensitive to the specific theory of radiation reaction, being in general non-zero. We illustrate this fact with the application two several models of radiation reaction.

Δ_q^μ could be observed with current or near future technology. We illustrate this in the case of a magnetic and linear field, when we apply it to a newly proposed theory of higher order electrodynamics [5, 6].

\mathcal{P} , \mathcal{T} and \mathcal{C} transformations in Maxwell's electromagnetic theory

Let the spacetime be described by a Lorentzian manifold (M, η) , where M is a four dimensional manifold. In a linear, homogeneous electromagnetic media, Maxwell's equations can be casted in a coordinate invariant form,

$$dF = 0, \quad d \star F = J, \quad (1)$$

where F is the Faraday 2-form, \star is the Hodge's star operator and J is the charge current density. We consider a timelike observer W and the orthogonal complement of W with respect to η by Π_W . The Maxwell's equations in the corresponding 1 + 3 decomposition in natural units are

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (2)$$

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$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} = (\vec{J} + \frac{\partial \vec{E}}{\partial t}), \quad (3)$$

The operations \mathcal{P} , \mathcal{T} are defined by the endomorphisms

$$\mathcal{P} : \Gamma TM \rightarrow \Gamma TM, \quad P(W) = W, \quad P(Z) = -Z, \quad \forall Z \in \Gamma \Pi_W, \quad (4)$$

$$\mathcal{T} : \Gamma TM \rightarrow \Gamma TM, \quad P(W) = -W, \quad T(Z) = Z, \quad \forall Z \in \Gamma \Pi_W. \quad (5)$$

The transformation \mathcal{C} is partially defined by its action on the charge density current,

$$\mathcal{C}(J) = -J. \quad (6)$$

\mathcal{C} is not a linear endomorphism of ΓTM . Indeed, we will define its action for each of the cases that we will encounter.

The way the operations \mathcal{P} , \mathcal{T} and \mathcal{C} act on $(\vec{E}, \vec{B}, \rho, \vec{J})$ is dictated by the assumptions that Maxwell's equations are invariant under the above transformations. A solution for this problem is given in *Table 1*,

Table 1: \mathcal{P} , \mathcal{T} and \mathcal{C} transformation rules 1

	\mathcal{P}	\mathcal{T}	\mathcal{C}
$\vec{\nabla}$	$-\vec{\nabla}$	$\vec{\nabla}$	$\vec{\nabla}$
∂_t	∂_t	$-\partial_t$	∂_t
\vec{E}	$-\vec{E}$	\vec{E}	$-\vec{E}$
\vec{B}	\vec{B}	$-\vec{B}$	$-\vec{B}$
ρ	ρ	ρ	$-\rho$
\vec{J}	$-\vec{J}$	$-\vec{J}$	$-\vec{J}$

With such transformation rules, equations (2) and (3) are invariant for each of the transformations \mathcal{P} , \mathcal{T} and \mathcal{C} independently.

The action of \mathcal{P} , \mathcal{T} and \mathcal{C} on the equations of motion of a point charged particle

Let us consider the Lorentz force equation written in the 1+3 decomposition,

$$m \ddot{x} = q \gamma (\vec{E} + \dot{x} \times \vec{B}), \quad (7)$$

where m is the rest mass of the particle, q is the charge and γ is the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\dot{x}^2}}$. As before, we can write down the transformation rule for (m, q) under \mathcal{P} , \mathcal{T} and \mathcal{C} , that are compiled as definition in *Table 2*,

Table 2: \mathcal{P} , \mathcal{T} and \mathcal{C} transformation rules 2

	\mathcal{P}	\mathcal{T}	\mathcal{C}
m	m	m	m
q	q	q	$-q$
$\frac{d}{d\tau}$	$\frac{d}{d\tau}$	$-\frac{d}{d\tau}$	$\frac{d}{d\tau}$

The direct consequence is that the Lorentz force equation (7) is invariant under the action of \mathcal{P} , \mathcal{T} and \mathcal{C} , given by *Table 1*, *Table 2*. Also, the four dimensional expression of the Lorentz force equation

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu, \quad \mu, \nu = 0, 1, 2, 3 \quad (8)$$

is also invariant. Here the time derivatives are taken with respect to τ , where τ is the proper-time parameter associated to $x : I \rightarrow M$ by η . The action of \mathcal{P} , \mathcal{T} and \mathcal{C} on τ and $\frac{d}{d\tau}$ is given in *Table 2*. Note that we did not consider the transformation of the initial conditions under \mathcal{P} , \mathcal{T} and \mathcal{C} .

If \ddot{x}_q is the acceleration appearing in the Lorentz force law under the external \vec{E} and \vec{B} of a particle of mass and charge (m, q) and \ddot{x}_{-q} is the acceleration for a particle $(m, -q)$ under the same external fields \vec{E} and \vec{B} , one has the relation

$$\Delta_q^\mu(x) := \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu = 0. \quad (9)$$

For the Lorentz force equation, this *symmetrized four-acceleration* is invariant under \mathcal{P} , \mathcal{T} and \mathcal{C} independently. The physical meaning of the relation (9) is that for a given external electromagnetic field described by the Faraday 2-form F , if a particle of mass and charge (m, q) has as world-line a solution of the Lorentz force equation (8) with acceleration \ddot{x}_q , then the particle with $(m, -q)$ will follow a world-line with exactly opposite acceleration $\ddot{x}_{-q} = -\ddot{x}_q$. Note that the external electric and magnetic fields are the same for both particles with (m, q) and $(m - q)$.

The Lorentz force equation does not take into account the reaction due to the emission of radiation. Many equations has been proposed in the literature to solve this problem of the radiation reaction (for very informative source of the different approaches the reader can consult for instance [4, 12, 15]). Therefore, it is natural to ask how \mathcal{P} , \mathcal{T} and \mathcal{C} act on these equations. We will restrict our attention to the case of the Lorentz-Dirac equation [3], the Landau-Lifshitz equation [8, 14, 15], the Bonnor-Larmor equations [1, 9] and a newer equation of motion obtained in the framework of a theory of higher order fields [5, 6].

The Lorentz-Dirac equation [3] is given by

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu + \frac{2}{3} q^2 (\ddot{x}^\mu - (\ddot{x}^\rho \ddot{x}_\rho) \dot{x}^\mu), \quad \ddot{x}^\mu \ddot{x}_\mu := \ddot{x}^\mu \ddot{x}^\sigma \eta_{\mu\sigma}. \quad (10)$$

It is easy to check that the Lorentz-Dirac equation is not invariant under \mathcal{P} . It is also not invariant under \mathcal{T} . Indeed, under time inversion \mathcal{T} , equation (10) is transformed to

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu - \frac{2}{3} q^2 (\ddot{x}^\mu - (\ddot{x}^\rho \ddot{x}_\rho) \dot{x}^\mu), \quad \ddot{x}^\mu \ddot{x}_\mu := \ddot{x}^\mu \ddot{x}^\sigma \eta_{\mu\sigma}.$$

The Lorentz-Dirac equation is invariant under the $\mathcal{CP}\mathcal{T}$ transformation.

The corresponding symmetrized four-acceleration analogue of equation (9) for the Lorentz-Dirac equation is

$$\Delta_q^\mu(x) = \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu = \frac{4}{3} q^2 (\ddot{x}^\mu - (\ddot{x}^\rho \ddot{x}_\rho) \dot{x}^\mu). \quad (11)$$

This symmetric combination is invariant under \mathcal{C} , \mathcal{CP} and under $\mathcal{CP}\mathcal{T}$ operation. It is not invariant under the actions of \mathcal{P} and \mathcal{T} alone.

Since the Landau-Lifshitz equation is a reduction of order of the Lorentz-Dirac equation, it is not a surprise that the Landau-Lifshitz equation has the same properties under \mathcal{P} , \mathcal{T} and \mathcal{C} than the Lorentz-Dirac equation. In four dimensional language, the Landau-Lifshitz equation is written as [11]

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu + \frac{2}{3} q^2 \left(\frac{q}{m} \dot{x}^\sigma (\partial_\sigma F_{\mu\nu}) \dot{x}^\nu + \frac{q^2}{m^2} F^{\mu\alpha} F_{\alpha\nu} \dot{x}^\nu - \frac{q^2}{m^2} (F_\sigma^\alpha \dot{x}^\sigma) (F_{\alpha\nu} \dot{x}^\nu) \dot{x}^\mu \right). \quad (12)$$

This equation is not invariant under \mathcal{P} , as it is shown after a short check. Under \mathcal{T} , the Landau-Lifshitz equation is not invariant and indeed the action of \mathcal{T} in equation (12) gives the expression

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu + \frac{2}{3} q^2 \left(\frac{q}{m} \dot{x}^\sigma (\partial_\sigma F_{\mu\nu}) \dot{x}^\nu - \frac{q^2}{m^2} F^{\mu\alpha} F_{\alpha\nu} \dot{x}^\nu + \frac{q^2}{m^2} (F_\sigma^\alpha \dot{x}^\sigma) (F_{\alpha\nu} \dot{x}^\nu) \dot{x}^\mu \right).$$

A similar expression is obtained after the action of the charge conjugation operator \mathcal{C} . However, the Landau-Lifshitz equation is invariant under the combined action $\mathcal{CP}\mathcal{T}$.

The symmetrized four-acceleration is in this case

$$\Delta_q^\mu(x) = \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu = \frac{4}{3} q^2 \frac{q^2}{m^2} (F^{\mu\alpha} F_{\alpha\nu} \dot{x}^\nu - (F_\sigma^\alpha \dot{x}^\sigma) (F_{\alpha\nu} \dot{x}^\nu) \dot{x}^\mu). \quad (13)$$

This symmetric combination is invariant under \mathcal{C} , \mathcal{PT} and $\mathcal{CP}\mathcal{T}$.

Let us consider now the equations of motion in Bonnor's theory [1], based on a previous idea of Larmor [9]. These are the combined equations of the form

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu \quad (14)$$

and

$$\frac{dm}{d\tau} = \frac{2}{3} q^2 \ddot{x}^\nu \ddot{x}_\nu. \quad (15)$$

The combination of both equations provides a dynamical system where the energy radiated by the point particle is originated in the change of the rest mass of the particle. Let us see the transformation properties of (14) and (15) separately.

Formally, equation (14) is the Lorentz force equation, but with a variable mass. Both equations (14) and (15) are invariant under parity inversion \mathcal{P} . However, Bonnor-Larmor theory cannot be invariant under the action of \mathcal{T} . This is in contrast with the Lorentz force equation. Indeed, by *Table 2*, if m is invariant under time inversion operation \mathcal{T} , then equation (15) cannot be consistent, since the right-hand side is invariant but the left side is not: after applying to (15) the \mathcal{T} operation, equation (15) transforms to

$$-\frac{dm}{d\tau} = \frac{2}{3}q^2 \ddot{x}^\nu \ddot{x}_\nu,$$

while the first Bonnor-Larmor equation (14) remains invariant². Under the action of \mathcal{C} , Bonnor-Larmor theory is invariant. The theory is not invariant by the action of \mathcal{CT} or \mathcal{CPT} .

For Bonnor-Larmor's theory, the symmetrized four-acceleration is

$$\Delta_q^\mu(x) = \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu = 0. \quad (16)$$

This is a qualitatively different prediction than for the Lorentz-Dirac and the Landau-Lifshitz theories, given by the relations (11) and (13) respectively, since in Bonnor-Larmor's theory the symmetrized acceleration is zero. Therefore, the symmetrized four-acceleration is a candidate for testing models of radiation reaction, since it is able to distinguish between theories of radiation reaction.

The last example that we will consider is a new equation of motion for point charged particles obtained in the framework of *higher order field electrodynamics* [5, 6],

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu - \frac{2}{3} q^2 \eta_{\nu\sigma} \ddot{x}^\nu \ddot{x}^\sigma \dot{x}^\mu. \quad (17)$$

As Bonnor's equation, (17) does not have pre-accelerated solutions and run-away solutions. In fact, equation (17) formally coincides with Bonnor's theory if m is variable as prescribed by Bonnor's equation (15). However, the theory presented in [5] and [6], the inertial mass m is constant and the further energy is taken from extra-terms in the definition of the higher order electromagnetic field. It can be seen that (17) is invariant under \mathcal{P} . However, it is not invariant under \mathcal{T} and \mathcal{C} . Indeed, by the action of \mathcal{T} , equation (17) transforms to

$$m \ddot{x}^\mu = q F^\mu{}_\nu \dot{x}^\nu + \frac{2}{3} q^2 (\ddot{x}^\nu \ddot{x}_\nu) \dot{x}^\mu$$

The same equation holds after the action of the time inversion \mathcal{C} . However, the equation is invariant by the action of the transformation \mathcal{CPT} .

The symmetrized four-acceleration is in this case given by the expression

$$\Delta_q^\mu(x) = \ddot{x}_q^\mu + \ddot{x}_{-q}^\mu = -\frac{4}{3}q^2 (\ddot{x}^\nu \ddot{x}_\nu) \dot{x}^\mu. \quad (18)$$

The relation (18) is invariant by the action of the action of \mathcal{C} , \mathcal{PT} and \mathcal{CPT} .

It was convenient to approximate

$$\ddot{x}^\mu = \frac{q}{m} F^\mu{}_\nu \dot{x}^\nu \quad (19)$$

in (17), thus obtaining

$$\ddot{x}^\mu = e \dot{x}^\nu F^\sigma{}_\nu \left(\frac{1}{m} \delta^\mu{}_\sigma - \frac{2e^2}{3m^2} \dot{x}^\mu F_{\sigma\lambda} \dot{x}^\lambda \right). \quad (20)$$

With this approximation, the equation of motion (18) is

$$\Delta_q^\mu(x) = -\frac{4}{3}q^2 \frac{q^2}{m^2} (F^\mu{}_\sigma \dot{x}^\sigma) (F_{\mu\lambda} \dot{x}^\lambda) \dot{x}^\mu. \quad (21)$$

This is invariant under the actions of \mathcal{C} , \mathcal{PT} and \mathcal{CPT} . The above results are collected in *Table 3*.

Table 3: Δ_q^μ for several models of point charged particle

	Lorentz	Lorentz-Dirac	Landau-Lifshitz	Bonnor-Larmor	Higher order fields
Δ_q^μ	0	$\frac{4}{3}q^2 (S_{LD}^\mu - S^\mu)$	$\frac{4}{3}q^2 \frac{q^2}{m^2} (T_{LL}^\mu)$	0	$-\frac{4}{3}q^2 S^\mu$

with

$$T_{LL}^\mu := (F^{\mu\alpha} F_{\alpha\nu} \dot{x}^\nu - (F_\sigma^\alpha \dot{x}^\sigma)(F_{\alpha\nu} \dot{x}^\nu) \dot{x}^\mu), \quad S_{LD}^\mu = \dot{x}^\mu, \quad S^\mu = (\ddot{x}^\rho \ddot{x}_\rho) \dot{x}^\mu.$$

²If instead one considers the transformation rule $\mathcal{T}(m) = -m$, equation (15) is invariant, but then equation (14) is not invariant.

Relevant examples of symmetrized four-acceleration.

In order to obtain how the symmetric acceleration $\Delta_q^\mu(x)$ is related with experimental settings, let us consider the $1 + 3$ decomposition and apply it to the relation given by the equation (21) for the model based on the equation (17). Thus, in the case of the equation (17) and after using the approximation (19), we have for the symmetric acceleration the expression

$$\vec{\Delta}_q(\vec{x}) = 2 \frac{\epsilon}{c^2} \frac{q^2}{m^2} \gamma^2 (\vec{E} + \dot{\vec{x}} \times \vec{B})^2 \dot{\vec{x}}, \quad (22)$$

where c is the speed of light and $\epsilon = \frac{2}{3} \frac{q^2}{mc^3}$ is the *characteristic time* associated with the classical radius of a point charged particle [12, 15]. For an electron or positron it has the value $\epsilon = 0.62 \times 10^{-23} \text{s}$ and for more massive particles, the corresponding ϵ could be even smaller. Therefore, we will consider electrons and positrons in our considerations below.

There are two ideal situations of special relevance:

There is only magnetic field. In this case, we have an expression of the form

$$|\vec{\Delta}_q(\vec{x})|(\vec{0}, \vec{B}) = 2 \frac{\epsilon}{c^2} \frac{q^2}{m^2} \gamma^2 (\dot{\vec{x}} \times \vec{B})^2 |\dot{\vec{x}}|.$$

The ratio with the absolute value of the Lorentz acceleration three vector is

$$\mathcal{R}_q(\vec{0}, \vec{B}) := \frac{|\vec{\Delta}_q(\vec{x})|}{|\dot{\vec{x}}|} = 2 \frac{\epsilon}{c^2} \frac{q}{m} \gamma |\dot{\vec{x}} \times \vec{B}| |\dot{\vec{x}}| + \mathcal{O}(\epsilon^2), \quad (23)$$

where we have use the fact that the radiation reaction term is smaller compared with the Lorentz force term. We can figure out the order of magnitude for the adimensional quantity (23) that is available with current technology. For this, a further simplification can be done if $c = |\dot{\vec{x}}|$, that corresponds to the ultra-relativistic regime. For a charged particle $q = e$ and if $|\dot{\vec{x}} \times \vec{B}| = |\dot{\vec{x}}| |\vec{B}| \sin \theta$, one has at leading order in ϵ

$$\mathcal{R}_q(\vec{0}, \vec{B}) = 2.17 \times 10^{-12} \gamma \sin \theta |\vec{B}| (\text{Tesla}), \quad (24)$$

where in this a-dimensional expression, the magnetic field is expressed in Tesla. For current electron accelerators, the factor γ can be of order 10^2 , while the magnetic fields achievable in the magnets is of order 1 to 10 Tesla [13]. This provides a range for the factor with maximal values of the order $\mathcal{R}_q(\vec{0}, \vec{B}) \simeq 10^{-10} - 10^{-9}$. A small difference in the behavior of electrons and positrons interacting with an external magnetic field \vec{B} .

There is only electric field. In this case, the leading order in ϵ gives

$$\vec{\Delta}_q(\vec{x})(\vec{E}, \vec{0}) = 2 \frac{\epsilon}{c^2} \frac{q^2}{m^2} \gamma^2 \vec{E}^2 \dot{\vec{x}}.$$

Taking the ratio with the module of the vector acceleration, one obtains

$$\mathcal{R}_q(\vec{E}, \vec{0}) = 2 \frac{\epsilon}{c^2} \frac{q}{m} \gamma |\vec{E}| |\dot{\vec{x}}|. \quad (25)$$

For an electron, if we write $c = |\dot{\vec{x}}|$ we find that equation (25) is

$$\mathcal{R}_q(\vec{E}, \vec{0}) = 0.73 \times 10^{-20} \gamma |\vec{E}| (\text{Newton} \cdot \text{Coulomb}^{-1}), \quad (26)$$

where in this adimensional expression, the electric field is expressed in $\text{Newton} \cdot \text{Coulomb}^{-1}$.

Current technology in laser-plasma acceleration provides electric fields of order 1 GeV/cm and energies of 1 GeV or higher for electron in plasma acceleration [7, 10]. Thus for such fields and kinematical conditions, one has that $\mathcal{R}_q(\vec{E}, \vec{0}) \simeq 0.73 \times 10^{-7}$, that although small, it could be observable with current or future laser-plasma acceleration technology.

Similar consequences can be drawn for others equations of motion of a point charged particle, as we have depicted in *Table 3*. One expects that T_{LL} is of the same order than $S_{LD} - S$. For the Lorentz-Dirac and equation (17), Δ_q^μ coincide for static fields. However, they will give different results in the non-static case, promising an avenue to experimentally detect the dynamical effects of the Schott term. The qualitative different behavior for the Lorentz-Dirac, Landau-Lifshitz and the equation (17) from one side and the Lorentz and Bonnor theories is clear.

From the experimental point of view, a major difficulty is to consider positron plasmas systems with the required specifications.

Conclusion

We have seen that the analysis of the symmetrized acceleration provides a tool to investigate the qualitative behavior of models for radiation reaction of point particles that is measurable with the current accelerator technology. We note that the effects are amplified by the relativistic gamma factor γ . Thus, we should expect indications of the effect along with the increase in energy.

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